

MATH 212

Assignment 4

3.2.1 (d-h) - 3.2.2 (b-d) - 3.2.3 - 3.2.4 - 3.2.5 -

3.2.6 (a-c) - 3.2.15 (a-d) - 3.2.18 - 3.2.27 (a) -

3.2.31 (a, b, d, e) - 3.2.34 - 3.2.41 (a-b) - 3.2.43

3.2.1 - d - $f(x) = x^2$

12 The Fourier series of f defined on $-\pi \leq x \leq \pi$

is $\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$ where:

$$a_0 = \langle f, 1 \rangle = \langle x^2, 1 \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{1}{\pi} \left[\frac{x^3}{3} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left[\frac{\pi^3}{3} - \frac{(-\pi)^3}{3} \right] = \frac{2}{3} \pi^2 \quad 1$$

$$a_k = \langle f, \cos kx \rangle = \langle x^2, \cos kx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos kx dx$$

$$= \frac{1}{\pi} \left[\frac{1}{k} x^2 \sin kx + \frac{2x}{k^2} \cos kx - \frac{2}{k^3} \sin kx \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left[0 + \frac{2\pi}{k^2} \cos k\pi + \frac{2\pi}{k^2} \cos k\pi \right]$$

$$= \frac{4}{k^2} \cos k\pi = \frac{4}{k^2} (-1)^k = \frac{(-1)^k 4}{k^2} \quad 2$$

x^2	\searrow	$\cos kx$
$2x$	\searrow	$\frac{1}{k} \sin kx$
2	\searrow	$-\frac{1}{k^2} \cos kx$
0	\searrow	$-\frac{1}{k^3} \sin kx$

$$b_k = \langle f, \sin kx \rangle = \langle x^2, \sin kx \rangle$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \sin kx dx$$

$$= \frac{1}{\pi} \left[-\frac{x^2}{k} \cos kx + \frac{2x}{k^2} \sin kx + \frac{2}{k^3} \cos kx \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left[-\frac{\pi^2}{k} \cos k\pi + 0 + \frac{2}{k^3} \cos k\pi + \frac{\pi^2}{k} \cos k\pi + 0 - \frac{2}{k^3} \cos k\pi \right]$$

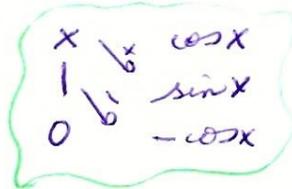
$$= 0$$

$$F_s(f(x)) = \frac{\pi^2}{3} + \sum_{k=1}^{\infty} \frac{(-1)^k 4}{k^2} \cos kx \quad 2$$

$$h - x \cos x$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos x \, dx$$

$$= \frac{1}{\pi} \left[x \sin x + \cos x \right]_{-\pi}^{\pi} = 0 \quad 1$$



$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos x \cos kx \, dx = 0$$

because $f(x) = x \cos x \cos kx$ is an odd function and the limits of the integral are symmetric. 2

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos x \sin kx \, dx$$

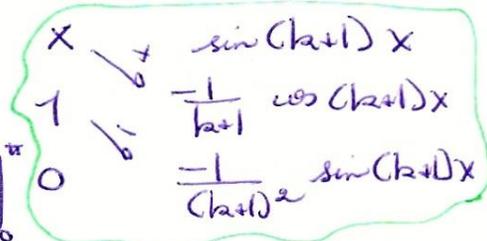
$$= \frac{2}{\pi} \int_0^{\pi} x \cos x \sin kx \, dx$$

$$= \frac{2}{\pi} \left[\int_0^{\pi} \frac{x \sin(kx+x)}{2} \, dx + \int_0^{\pi} \frac{x \sin(kx-x)}{2} \, dx \right] \quad 2$$

$$= \frac{1}{\pi} \left[\int_0^{\pi} x \sin(k+1)x \, dx + \int_0^{\pi} x \sin(k-1)x \, dx \right]$$

$$= \frac{1}{\pi} \left[\frac{-x}{k+1} \cos(k+1)x + \frac{1}{(k+1)^2} \sin(k+1)x \right]_0^{\pi}$$

$$+ \frac{1}{\pi} \left[\frac{-x}{k-1} \cos(k-1)x + \frac{1}{(k-1)^2} \sin(k-1)x \right]_0^{\pi}$$



$$= \frac{1}{\pi} \left[\frac{-\pi}{k+1} \cos(k+1)\pi + 0 + 0 + 0 \right]$$

$$+ \frac{1}{\pi} \left[\frac{-\pi}{k-1} \cos(k-1)\pi + 0 + 0 + 0 \right]$$

$$= \frac{-\cos(k+1)\pi}{k+1} - \frac{\cos(k-1)\pi}{k-1} = \frac{(-1)^k}{k+1} + \frac{(-1)^k}{k-1}$$

$$= (-1)^k \left[\frac{1}{k+1} + \frac{1}{k-1} \right] = \frac{(-1)^k 2k}{(k-1)(k+1)} \quad 4$$

$$F_s(f(x)) = \sum_{k=1}^{\infty} \frac{(-1)^k 2k}{(k-1)(k+1)} \sin kx$$

$$2.2- b- f(x) = \begin{cases} 1 & \frac{1}{2}\pi < |x| < \pi \\ 0 & \text{otherwise} \end{cases}$$

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$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \left[\int_{-\pi}^{-\frac{1}{2}\pi} dx + \int_{\frac{1}{2}\pi}^{\pi} dx \right]$$

$$= \frac{1}{\pi} \left[-\frac{1}{2}\pi + \pi + \pi - \frac{1}{2}\pi \right]$$

$$= \frac{1}{\pi} [2\pi - \pi] = \frac{\pi}{\pi} = 1 \quad 1$$

$$a_{ka} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx = \frac{1}{\pi} \left[\int_{-\pi}^{-\frac{1}{2}\pi} \cos kx dx + \int_{\frac{1}{2}\pi}^{\pi} \cos kx dx \right]$$

$$= \frac{1}{\pi} \left[\frac{1}{k} \sin kx \right]_{-\pi}^{-\frac{1}{2}\pi} + \frac{1}{\pi} \left[\frac{1}{k} \sin kx \right]_{\frac{1}{2}\pi}^{\pi}$$

$$= \frac{1}{\pi} \left[\frac{1}{k} \sin \frac{k\pi}{2} + \frac{1}{k} \sin k\pi + \frac{1}{k} \sin k\pi - \frac{1}{k} \sin \frac{k\pi}{2} \right]$$

$$= \frac{-2}{k\pi} \sin \frac{k\pi}{2} = \begin{cases} 0 & k \text{ even} \\ \frac{2(-1)^j}{(2j-1)\pi} & k \text{ odd, } k=2j-1 \end{cases} \quad 2$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx dx = \frac{1}{\pi} \left[\int_{-\pi}^{-\frac{1}{2}\pi} \sin kx dx + \int_{\frac{1}{2}\pi}^{\pi} \sin kx dx \right]$$

$$= \frac{1}{\pi} \left[-\frac{1}{k} \cos kx \right]_{-\pi}^{-\frac{\pi}{2}} + \frac{1}{\pi} \left[-\frac{1}{k} \cos kx \right]_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{1}{\pi} \left[0 + \frac{\cos k\pi}{k} - \frac{\cos k\pi}{k} + 0 \right] = 0 \quad 2$$

$$\tilde{f}_s(f(x)) = \frac{1}{2} + \sum_{j=1}^{\infty} \frac{2(-1)^j}{(2j-1)\pi} \cos(2j-1)x$$

$$d- f(x) = \begin{cases} x & |x| < \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases} = \begin{cases} x & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 0 & \text{---} \end{cases}$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x dx \right] = \frac{1}{\pi} \cdot \left. \frac{x^2}{2} \right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{1}{\pi} \left[\frac{\pi^2}{8} - \frac{\pi^2}{8} \right] = 0$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x \cos kx dx = 0 \quad \text{2}$$

↑
Odd function

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx dx = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x \sin kx dx$$

$$= \frac{2}{\pi} \int_0^{\frac{\pi}{2}} x \sin kx dx$$

↑
even
 f^n

$$\begin{array}{l} x \cdot x \cdot \sin kx \\ 1 \cdot \int -\frac{1}{k} \cos kx \\ 0 \cdot -\frac{1}{k^2} \sin kx \end{array}$$

$$= \frac{2}{\pi} \left[-\frac{x}{k} \cos kx + \frac{1}{k^2} \sin kx \right]_0^{\frac{\pi}{2}} = \frac{2}{\pi} \left[-\frac{\frac{\pi}{2}}{2k} \cos \frac{k\pi}{2} + \frac{1}{k^2} \sin \frac{k\pi}{2} \right] = \begin{cases} -\frac{1}{k} \cos \frac{k\pi}{2} & \text{b. even} \\ \frac{1}{k^2} \sin \frac{k\pi}{2} & \text{b. odd} \end{cases}$$

$$= \begin{cases} -\frac{1}{2^j} \cos(j\pi) & j=0, 2, 4, \dots \\ \frac{1}{(2j+1)^2} \sin(2j+1)\frac{\pi}{2} & j=0, 1, 2, \dots \end{cases} = \begin{cases} -\frac{1}{2^j} (-1)^j & j=0, 1, 2, \dots \\ \frac{1}{(2j+1)^2} (-1)^j & j=0, 1, 2, \dots \end{cases}$$

$$FSC[f(x)] = \sum_{j=0}^{\infty} \frac{(-1)^{j+1}}{2^j} \sin 2jx + \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)^2} \sin(2j+1)x$$

3.2.3- Since $\cos 2x = 1 - 2\sin^2 x$, then $\sin^2 x = \frac{1}{2} - \frac{\cos 2x}{2}$ 1/2

3 Since $\cos 2x = 2\cos^2 x - 1$, then $\cos^2 x = \frac{1}{2} + \frac{\cos 2x}{2}$ 1/2

3.2.4- 8 $g(x) = \frac{1}{2} P_0 + \sum_{k=1}^{\infty} (P_k \cos kx + Q_k \sin kx)$

$$a_l = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \cos lx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \left[\frac{1}{2} P_0 \cos lx \right] dx + \frac{1}{\pi} \int_{-\pi}^{\pi} \left[\sum_{k=1}^{\infty} P_k \cos kx \cos lx + Q_k \sin kx \cos lx \right] dx$$

$$= \frac{1}{2} P_0 \cdot \left. \frac{1}{l} \sin lx \right|_{-\pi}^{\pi} + \frac{1}{\pi} [P_l \pi]$$

$$= \frac{P_0}{2l} [0+0] + P_l = P_l \quad 0 < l \leq m$$

Similarly $b_l = Q_l$ for $0 < l \leq m$

④

However, when $l > m$,

$$\int_{-\pi}^{\pi} \left[\sum_{k=1}^{\infty} P_k \cos kx \cos lx + Q_k \sin kx \cos lx \right] dx = 0$$

Hence $a_0 = 0$

Similarly $b_0 = 0$.

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2} P_0 + \sum_{k=1}^m [P_k \cos kx + Q_k \sin kx] dx$$

$$= \frac{1}{\pi} \left[\frac{1}{2} P_0 x \right]_{-\pi}^{\pi} + \frac{1}{\pi} \sum_{k=1}^m \int_{-\pi}^{\pi} P_k \cos kx dx$$

$$+ \frac{1}{\pi} \sum_{k=1}^m \int_{-\pi}^{\pi} \underbrace{Q_k \sin kx}_{\text{odd}} dx$$

$$= P_0 + \frac{1}{\pi} \sum_{k=1}^m \left[-\frac{P_k}{k} \sin kx \right]_{-\pi}^{\pi}$$

$$= P_0 + 0 = P_0$$

3.2.5. a. True

4 b. False

c. True

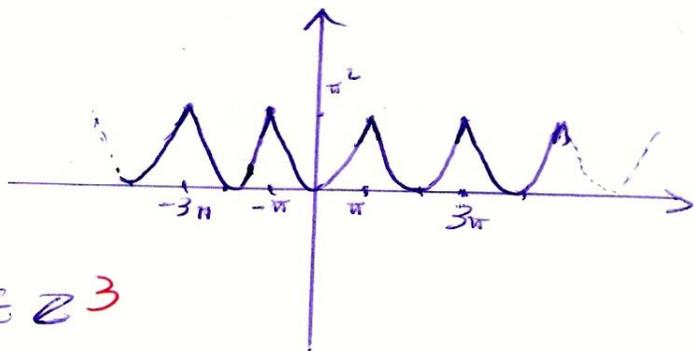
d. False.

3.2.6 - a. $f(x) = x^2$

6 $\tilde{f}(x)$ is continuous

$\tilde{f}(x)$ is not diff

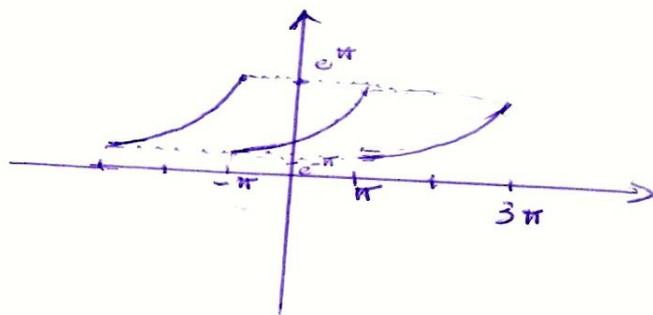
at $x = (2k+1)\pi, k \in \mathbb{Z}$



c. $f(x) = e^x$

$\tilde{f}(x)$ is not cts nor

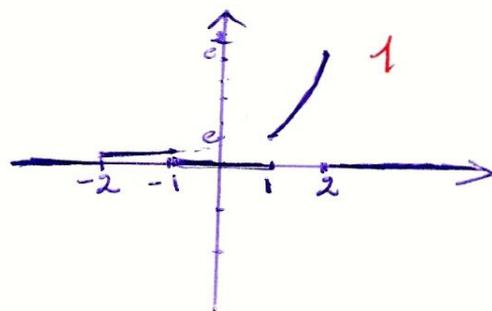
diff at $x = (2k+1)\pi, k \in \mathbb{Z}$



3.2.15 - a. $f(x) = \begin{cases} e^x & 1 < |x| < 2 \\ 0 & \text{otherwise} \end{cases}$

$= \begin{cases} e^x & 1 < x < 2 \\ -2 < x < -1 \\ 0 & \text{otherwise} \end{cases}$

jump discont	jump magnitude
$x_1 = -2$	$f(-2^+) - f(-2^-) = e^{-2}$
$x_2 = -1$	$f(-1^+) - f(-1^-) = -e^{-1}$
$x_3 = 1$	$f(1^+) - f(1^-) = e$
$x_4 = 2$	$f(2^+) - f(2^-) = -e^2$



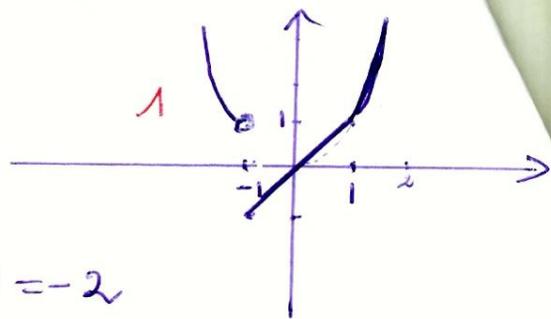
$$d) f(x) = \begin{cases} x & |x| \leq 1 \\ x^2 & |x| > 1 \end{cases}$$

jump
disk

$$1 \quad x = -1$$

jump
mold

$$f(-1+) - f(-1-) = -1 - 1 = -2$$



$$3.2.18- f(x) = x^{\frac{1}{3}}$$

4.5

Is f piecewise cts? Yes $1^{\frac{1}{2}}$

Is $f \in C^1$ piecewise? No bcs $\lim_{x \rightarrow 0^+} f'(x) = +\infty$ $1^{\frac{1}{2}}$

Is $f \in C^2$ piecewise? No (same justification) $1^{\frac{1}{2}}$

$$3.27-a- f(x) = e^x$$

1.1

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x dx = \frac{1}{\pi} [e^x]_{-\pi}^{\pi} = \frac{1}{\pi} [e^{\pi} - e^{-\pi}]$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \cos kx dx$$

$$u = e^x \quad v' = \cos kx$$

$$u' = e^x \quad v = \frac{1}{k} \sin kx$$

$$a_k = \frac{1}{\pi} \left[\frac{1}{k} e^x \sin kx \right]_{-\pi}^{\pi} - \frac{1}{\pi k} \int_{-\pi}^{\pi} e^x \sin kx dx$$

$$u = e^x \quad v' = -\sin kx$$

$$u' = e^x \quad v = \frac{1}{k} \cos kx$$

$$a_k = \frac{1}{\pi} \left[\frac{1}{k} e^x \sin kx \right]_{-\pi}^{\pi} + \frac{1}{\pi k} \left[e^x \cdot \frac{1}{k} \cos kx \right]_{-\pi}^{\pi}$$

$$- \frac{1}{\pi k^2} \int_{-\pi}^{\pi} e^x \cos kx dx$$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} e^x \cos kx dx = \frac{1}{\pi k} \left[\frac{e^{\pi} \cos \pi k}{k} - \frac{e^{-\pi} \cos \pi k}{k} \right] - \frac{1}{\pi k^2} \int_{-\pi}^{\pi} e^x \cos kx dx$$

$$\left(\frac{1}{\pi} + \frac{1}{\pi k^2} \right) \int_{-\pi}^{\pi} e^x \cos kx dx = \frac{\cos \pi k}{\pi k^2} [e^{\pi} - e^{-\pi}]$$

6

$$\int_{-\pi}^{\pi} e^x \cos kx dx = \frac{\cos \pi k}{k^2 + 1} [e^{\pi} - e^{-\pi}]$$

$$a_k = \frac{e^{\pi} - e^{-\pi}}{\pi(k^2 + 1)} (-1)^k \quad \therefore \text{Similarly, } b_k = \frac{(e^{\pi} - e^{-\pi}) k (-1)^{k+1}}{\pi(k^2 + 1)}$$

$$\tilde{f}_s(e^x) = a_0 + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx, \text{ where } a_0, a_k \text{ and } b_k \text{ are written above.}$$

6

2.2.31 - a- $f(x) = x^2$
 $f(-x) = (-x)^2 = x^2 = f(x)$ even
6 $1\frac{1}{2}$

b- $f(x) = e^x$
 $f(-x) = e^{-x}$
 $f(-x) \neq f(x)$ not even
 $f(-x) \neq -f(x)$ not odd $1\frac{1}{2}$

d- $f(x) = \sin \pi x$
 $f(-x) = -\sin \pi x = -f(x)$ odd
 $1\frac{1}{2}$

e- $f(x) = \frac{1}{x}$
 $f(-x) = -\frac{1}{x} = -f(x)$ odd
 $1\frac{1}{2}$

3.2.34- If $f(x)$ is odd, then $f'(x)$ is even

6 Proof: f odd $\Rightarrow f(-x) = -f(x)$

$$\Rightarrow [f(-x)]' = -f'(x)$$

$$\Rightarrow -f'(-x) = -f'(x)$$

$$\Rightarrow f'(-x) = f'(x)$$

$$\Rightarrow f' \text{ is even.}$$

3.2.43- 11^{1/2} a- $f(x) = \begin{cases} \sin x & \text{if } \sin x \geq 0 \\ -\sin x & \text{if } -\sin x \leq 0 \end{cases}$ $f(x) = |\sin x|$ is even $\Rightarrow b_n = 0$.

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \left[\int_{-\pi}^0 -\sin x dx + \int_0^{\pi} \sin x dx \right]$$

$$= \frac{1}{\pi} \left[[\cos x]_{-\pi}^0 - [\cos x]_0^{\pi} \right] = \frac{1}{\pi} [1+1+1+1] = \frac{4}{\pi}^1$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx = \frac{1}{\pi} \left[\int_{-\pi}^0 -\sin x \cos kx dx + \int_0^{\pi} \sin x \cos kx dx \right]$$

$$= \frac{1}{\pi} \left[2 \int_0^{\pi} \sin x \cos kx dx \right]$$

$$= \frac{2}{\pi} \left[\int_0^{\pi} \frac{\sin(x+kx)}{2} dx + \int_0^{\pi} \frac{\sin(x-kx)}{2} dx \right]$$

$$= \frac{2}{\pi} \left[\int_0^{\pi} \frac{\sin(k+1)x}{2} dx + \int_0^{\pi} \frac{\sin(1-k)x}{2} dx \right]$$

If $k=1$, get $a_1 = \frac{2}{\pi} \int_0^{\pi} \frac{\sin 2x}{2} dx = \frac{1}{\pi} \left[-\frac{1}{2} \cos 2x \right]_0^{\pi} = \frac{1}{\pi} \left[-\frac{1}{2} + \frac{1}{2} \right] = 0$

If $k \neq 1$, get:

$$a_k = \frac{2}{\pi} \left[\frac{-1}{2(k+1)} \cos(k+1)x - \frac{1}{2(1-k)} \cos(1-k)x \right]_0^{\pi} \quad 6^{1/2}$$

$$= \frac{2}{\pi} \left[\frac{-1}{2(k+1)} \cos(k+1)\pi - \frac{1}{2(1-k)} \cos(1-k)\pi + \frac{1}{2(k+1)} + \frac{1}{2(1-k)} \right]$$

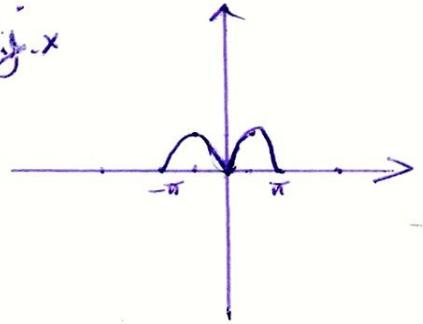
$$= \frac{2}{\pi} \left[\frac{-1}{2(k+1)} - \frac{1}{2(1-k)} + \frac{1}{2(k+1)} + \frac{1}{2(1-k)} \right] = 0 \quad \text{if } k \text{ is odd}$$

$$\left[\frac{2}{\pi} \left[\frac{1}{2(k+1)} + \frac{1}{2(1-k)} + \frac{1}{2(k+1)} + \frac{1}{2(1-k)} \right] \right] = \frac{2}{\pi} \left[\frac{1}{k+1} + \frac{1}{1-k} \right] \text{ if } k \text{ is even}$$

$$= \frac{4}{\pi} \cdot \frac{1}{(k+1)(1-k)}$$

Therefore:

$$\begin{aligned}\tilde{F}_S(\sin x) &= \frac{2}{\pi} + \sum_{j=1}^{\infty} \frac{-4}{\pi} \cdot \frac{1}{(2j)^2-1} \cdot \cos 2jx \\ &= \frac{2}{\pi} - \frac{4}{\pi} \sum_{j=1}^{\infty} \frac{\cos 2jx}{4j^2-1}\end{aligned}$$



b) Evaluate:

$$* \sum_{k=1}^{\infty} \frac{1}{4k^2-1}$$

First, the Fourier cosine series found above converges to the 2π -periodic extension of $|\sin x|$.

$$\begin{aligned}\therefore \text{For } x=0, \text{ we get } 0 &= \frac{2}{\pi} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{4k^2-1} \\ \Rightarrow \sum_{k=1}^{\infty} \frac{1}{4k^2-1} &= \frac{2}{\pi} \times \frac{\pi}{4} = \frac{1}{2} \quad 5\end{aligned}$$

$$* \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{4k^2-1}$$

$$\begin{aligned}\text{For } x=\frac{\pi}{2}, \text{ we get } 1 &= \frac{2}{\pi} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\cos(k\pi)}{4k^2-1} \\ &= \frac{2}{\pi} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{4k^2-1}\end{aligned}$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{4k^2-1} = \left(1 - \frac{2}{\pi}\right) \times \frac{\pi}{4} = \frac{\pi}{4} - \frac{1}{2} \quad 5$$